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# ON SOFIC SYSTEMS I

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ABSTRACT
Topological Markov chains are invariantly associated with sofic systems. A dimension function is introduced for sofic systems, and a criterion is given for a sofic system to be properly sofic.

## 1. Introduction

Let $\Sigma$ be a finite state space. We denote the shift on $\Sigma^{\mathbf{Z}}$ by $S_\Sigma$,

$$S_\Sigma((x_i)_{i\in\mathbf{Z}}) = (x_{i+1})_{i\in\mathbf{Z}}, \qquad (x_i)_{i\in\mathbf{Z}} \in \Sigma^{\mathbf{Z}}.$$

The dynamical system that is given by a closed $S_\Sigma$-invariant subset $Y$ of $\Sigma^{\mathbf{Z}}$ and by the restriction of $S_\Sigma$ to $Y$ will be denoted by $(Y, S_\Sigma)$. A zero-one transition matrix $(A(\sigma,\sigma'))_{\sigma,\sigma'\in\Sigma}$ determines a closed $S_\Sigma$-invariant subset $X_A$ of $\Sigma^{\mathbf{Z}}$,

$$X_A = \{(x_i)_{i\in\mathbf{Z}} \in \Sigma^{\mathbf{Z}} : A(x_i, x_{i+1}) = 1,\ i \in \mathbf{Z}\}.$$

The dynamical system $(X_A, S_\Sigma)$ is called a topological Markov chain. A dynamical system $(Y, S_\Sigma)$ is called sofic if it is the homomorphic image of a topological Markov chain. Sofic systems were first considered by B. Weiss [12, 13]. Since then they were studied by E. Coven and M. Paul [4, 5], R. Fischer [7, 8], B. Marcus [10], M. Boyle [3] and M. Nasu [11]. In this paper we attempt to elucidate further their structure.

To every sofic system there are associated topological Markov chains, that admit the sofic system as a homomorphic image of full entropy. Constructions of such chains have been given (see e.g. [4]). However, it seems not to have been noticed that some of these constructions are canonical. In section 2 we associate canonically to a sofic system $(Y, S_\Sigma)$ two topological Markov chains that we call the past and the future state chains of the system. Combining these two, we

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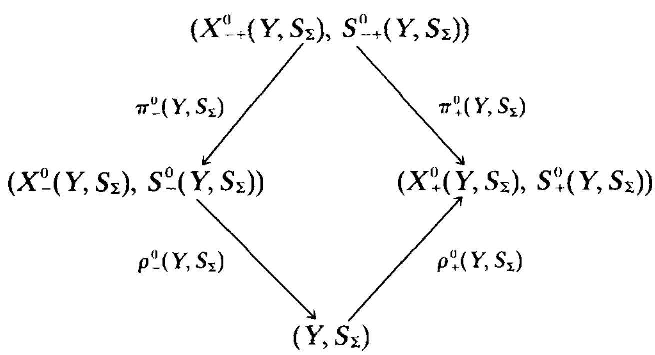
obtain another topological Markov chain canonically associated to $(Y, S_\Sigma)$ that we call the joint state chain of $(Y, S_\Sigma)$. Also, if $(Y, S_\Sigma)$ is topologically transitive and has periodic points dense, then we construct irreducible topological Markov chains that are canonical extensions of $(Y, S_\Sigma)$. We call these irreducible topological Markov chains the past and the future finitary state chains of $(Y, S_\Sigma)$, and denote them by $(X^0_-(Y, S_\Sigma), S^0_-(Y, S_\Sigma))$ and $(X^0_+(Y, S_\Sigma), S^0_+(Y, S_\Sigma))$. There will also be a joint finitary state chain associated to $(Y, S_\Sigma)$, denoted by $(X^0_{-+}(Y, S_\Sigma), S^0_{-+}(Y, S_\Sigma))$. Our methods are similar to the ones found in [3, 7, 8, 10, 13]. Compare here also the paper of I. Csiszar and J. Komlos [6]. The homomorphism of $(X^0_-(Y, S_\Sigma), S^0_-(Y, S_\Sigma))$ resp. of $(X^0_+(Y, S_\Sigma), S^0_+(Y, S_\Sigma))$ onto $(Y, S_\Sigma)$ will be denoted by $\rho^0_-(Y, S_\Sigma)$ resp. by $\rho^0_+(Y, S_\Sigma)$. There are also homomorphisms

$$\pi^0_-(Y, S_\Sigma) : (X^0_{-+}(Y, S_\Sigma),\ S^0_{-+}(Y, S_\Sigma)) \to (X^0_-(Y, S_\Sigma),\ S^0_-(Y, S_\Sigma)),$$

$$\pi^0_+(Y, S_\Sigma) : (X^0_{-+}(Y, S_\Sigma),\ S^0_{-+}(Y, S_\Sigma)) \to (X^0_+(Y, S_\Sigma),\ S^0_+(Y, S_\Sigma)).$$

Recall that a homomorphism is called right (left)-resolving if an inverse image of a point is uniquely determined by the point together with any initial (final) section of the inverse image. The homomorphisms $\pi^0_+(Y, S_\Sigma)$ and $\rho^0_-(Y, S_\Sigma)$ are left-resolving and the homomorphisms $\pi^0_-(\varphi, S_\Sigma)$ and $\rho^0_+(Y, S_\Sigma)$ are right-resolving. Thus the top half of the following commutative diagram is reminiscent of the situation encountered by R. Adler and B. Marcus in [1]:




The extensions that we construct for topologically transitive sofic systems with periodic points dense are canonical in the sense that, given two such sofic systems $(Y, S_\Sigma)$ and $(\bar{Y}, S_{\bar{\Sigma}})$ and a topological conjugacy

$$u : (Y, S_\Sigma) \to (\bar{Y}, S_{\bar{\Sigma}}),$$

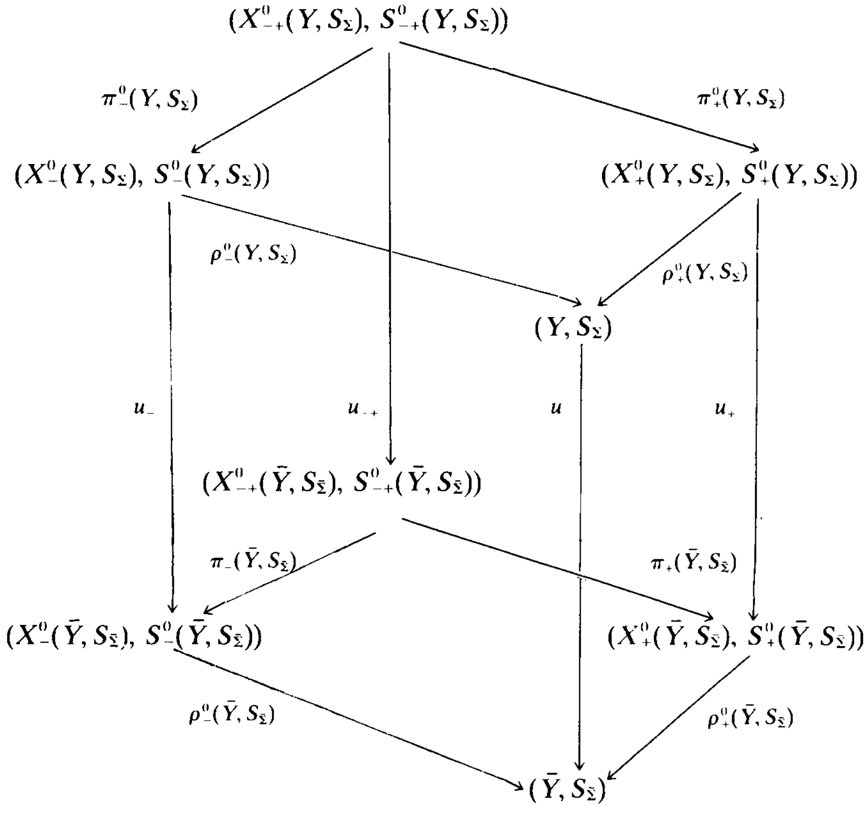
there exist unique topological conjugacies

$$u_- \;:(X^0_-(Y,S_\Sigma),\ S^0_-(Y,S_\Sigma))\to(X^0_-(\bar Y,S_{\bar\Sigma}),\ S^0_-(\bar Y,S_{\bar\Sigma})),$$

$$u_+:(X^0_+(Y,S_\Sigma),\ S^0_+(Y,S_\Sigma))\to(X^0_+(\bar Y,S_{\bar\Sigma}),\ S^0_+(\bar Y,S_{\bar\Sigma})),$$

$$u_{-+}:(X^0_{-+}(Y,S_\Sigma),\ S^0_{-+}(Y,S_\Sigma))\to(X^0_{-+}(\bar Y,S_{\bar\Sigma}),\ S^0_{-+}(\bar Y,S_{\bar\Sigma})),$$

such that the following diagram is commutative:




In [9] there were introduced past and future dimensions for topological Markov chains. In section 3 we extend these notions to sofic systems. We shall see that the range of the future dimension of a sofic system can be identified with the range of the dimension of its future state chain. In section 4 we apply this to give a criterion for a sofic system to be Markov. If $\varphi_{(Y,S_\Sigma)}$ is the automorphism that the sofic system $(Y,S_\Sigma)$ induces on the range of its future dimension, and if $\zeta_{(Y,S_\Sigma)}$ is the zeta-function of $(Y,S_\Sigma)$, then this result can be stated as follows: $(Y,S_\Sigma)$ is Markov if and only if

$$\zeta_{(Y,S_\Sigma)}(z)=\operatorname{Det}(1-z\varphi_{(Y,S_\Sigma)})^{-1}.$$

## 2. Some extensions of sofic systems

Consider again a finite state space $\Sigma$. We first indicate some notation to be used throughout this paper. For the projection of $\Sigma^{\mathbf{Z}}$ onto $\Sigma^{[j,k]}$, $j, k \in \mathbf{Z}$, $j \leqq k$, we write $P_{[j,k]}$, and for $P_{[j,k]}(x)$, $x \in \Sigma^{\mathbf{Z}}$, we write also $x_{[j,k]}$. Similar notation will also be used for other projections. For cylinder sets we employ notation of the following sort:

$$Z(\sigma) = \{(x_i)_{i \in \mathbf{Z}} \in \Sigma^{\mathbf{Z}} : x_1 = \sigma\}, \qquad \sigma \in \Sigma,$$

and

$$Z(a) = \{(x_i)_{i \in \mathbf{Z}} : x_{[j,k]} = a\}, \qquad a \in \Sigma^{[j,k]}, \quad j, k \in \mathbf{Z}, \quad j < k,$$

etc.

Let now $(Y, S_\Sigma)$ be a sofic system. We assign to every $x_- \in P_{(-\infty,i)}(Y)$ a closed subset $\omega_+(Y, S_\Sigma)(x_-)$ of $P_{[i,\infty)}(Y)$ by setting

$$\omega_+(Y, S_\Sigma)(x_-) = \{x_+ \in P_{[i,\infty)}(Y) : (x_-, x_+) \in Y\}, \qquad i \in \mathbf{Z}.$$

We are interested in the range of the mapping $\omega_+(Y, S_\Sigma)$. We set

$$\Xi_+(Y, S_\Sigma) = \omega_+(Y, S_\Sigma)(P_{(-\infty,0]}(Y)),$$

and also

$$\Xi_+^{(i)}(Y, S_\Sigma) = \omega_+(Y, S_\Sigma)(P_{(-\infty,i)}(Y)), \qquad i \in \mathbf{Z}.$$

In this way

$$\Xi_+^{(i)}(Y, S_\Sigma) = S_\Sigma^{-i+1} \Xi_+(Y, S_\Sigma), \qquad i \in \mathbf{Z}.$$

Let for a suitable finite state space $\hat{\Sigma}$ and zero-one transition matrix $(\hat{A}(\hat{\sigma}, \hat{\sigma}'))_{\hat{\sigma},\hat{\sigma}' \in \hat{\Sigma}}$, $(Y, S_\Sigma)$ be a homomorphic image of the topological Markov chain $(X_{\hat{A}}, S_{\hat{\Sigma}})$, the homomorphism being implemented by a one-block map $\Phi : \hat{\Sigma} \to \Sigma$. We denote then for $x_- \in P_{(-\infty,0]}(Y)$ by $\hat{\Delta}(x_-)$ the set of all $\hat{\sigma}_0 \in \hat{\Sigma}$ such that there is a

$$(\hat{\sigma}_i)_{-\infty < i \leqq 0} \in P_{(-\infty,0]}(X_{\hat{A}})$$

with

$$x_- = (\Phi(\hat{\sigma}_i))_{-\infty < i \leqq 0}.$$

It is

$$\omega_+(Y, S_\Sigma)(x_-) = \bigcup_{\hat{\sigma}_0 \in \hat{\Delta}(x_-)} \{(\Phi(\hat{\sigma}_i))_{1 \leqq i < \infty} : (\hat{\sigma}_i)_{1 \leqq i < \infty} \in P_{[1,\infty)}(X_{\hat{A}}), \hat{A}(\hat{\sigma}_0, \hat{\sigma}_1) = 1\},$$

(1)

$$x_- \in P_{(-\infty,0]}(Y).$$

(2.1) LEMMA. *$\Xi_+(Y, S_\Sigma)$ is finite.*

PROOF. It is seen from (1) that there are at most $|\hat{\Sigma}|!$ elements in $\Xi_+(Y, S_\Sigma)$.
Q.e.d.

We denote

$$\Omega_+(Y, S_\Sigma) = \{(\sigma, D) \in \Sigma \times \Xi_+(Y, S_\Sigma) : Z(\sigma) \cap D \neq \emptyset\},$$

and we define a transition matrix $A_+(Y, S_\Sigma)$ for the state space $\Omega_+(Y, S_\Sigma)$ by setting

$$A_+(Y, S_\Sigma)((\sigma, D), (\sigma', D')) = \begin{cases} 1 & \text{if } D' = S_\Sigma P_{[1,\infty)}(Z(\sigma) \cap D), \\ 0 & \text{elsewhere}, \end{cases}$$

$$(\sigma, D), (\sigma', D') \in \Omega_+(Y, S_\Sigma).$$

The topological Markov chain with state space $\Omega_+(Y, S_\Sigma)$ and transition matrix $A_+(Y, S_\Sigma)$ will be called the future state chain of $(Y, S_\Sigma)$. Instead of $(X_{A_+(Y, S_\Sigma)}, S_{\Omega_+(Y, S_\Sigma)})$ we write $(X_+(Y, S_\Sigma), S_+(Y, S_\Sigma))$. Note that a block

$$(\sigma_i, D_i)_{j \leqq i \leqq k} \in \Omega_+(Y, S_\Sigma)^{[j,k]}, \qquad j, k \in \mathbf{Z}, \quad j < k,$$

is admissible for $A_+(Y, S_\Sigma)$ if and only if one has that

$$(\sigma_i)_{j \leqq i \leqq k} \in P_{[j,k]}(Y),$$

and with

$$E_i = S_\Sigma^{-i+1} D_i, \qquad i \in \mathbf{Z},$$

that

$$E_i = P_{[i,\infty)}(Z((\sigma_l)_{j \leqq l \leqq i}) \cap E_j), \qquad j < i < k.$$

Further we have the following lemma.

(2.2) LEMMA. *Let*

$$(\sigma_i, D_i)_{i \in \mathbf{Z}} \in X_+(Y, S_\Sigma), \tag{2}$$

*and let*

$$E_i = S_\Sigma^{-i+1} D_i, \qquad i \in \mathbf{Z}.$$

*Then*

$$(\sigma_i)_{k \leqq i < \infty} \in E_k, \qquad k \in \mathbf{Z},$$

*and*

$$E_k \subset \omega_+(Y, S_\Sigma)((\sigma_i)_{-\infty < i < k}), \qquad k \in \mathbf{Z}.$$

PROOF. By (2) one has for all $j, k \in \mathbf{Z}$, $j < k$,

$$(\sigma_i)_{j \leqq i \leqq k} \in P_{[j,k]}(E_j)$$

and if

$$y_+ \in E_k$$

then one has by (2) that

$$((\sigma_j)_{j \leqq i \leqq k}, y_+) \in P_{[j,\infty)}(Y).$$

Use compactness arguments. Q.e.d.

We denote the projection that assigns to every $(\sigma_i, D_i)_{i \in \mathbf{Z}} \in X_+(Y, S_\Sigma)$ the point $(\sigma_i)_{i \in \mathbf{Z}} \in Y$ by $\rho_+(Y, S_\Sigma)$. This projection is in fact onto $Y$, and it has a Borel section $\tau_+(Y, S_\Sigma)$ that is given by

$$\tau_+(Y, S_\Sigma)(x) = (x_i, S_\Sigma^{i-1}\omega_+(Y, S_\Sigma)(x_{(-\infty,i)}))_{i \in \mathbf{Z}}, \qquad x \in Y.$$

To see this, observe that

$$\omega_+(Y, S_\Sigma)(x_{(-\infty,k)}) = P_{[k,\infty)}Z(x_{[j,k)}) \cap \omega_+(Y, S_\Sigma)(x_{(-\infty,j)}),$$

$$j, k \in \mathbf{Z}, \quad j < k, \quad x \in Y.$$

Also, note that $\rho_+(Y, S_\Sigma)$ is right resolving.

(2.3) LEMMA. *$\tau_+(Y, S_\Sigma)(Y)$ is dense in $X_+(Y, S_\Sigma)$.*

PROOF. Let $I \in \mathbf{N}$, and let $(\sigma_i, D_i)_{-I \leqq i \leqq I}$ be an $A_+(Y, S_\Sigma)$-admissible block. Setting

$$E_i = S_\Sigma^{-i+1} D_i, \qquad -I \leqq i \leqq I,$$

let

$$(\sigma_i)_{-\infty < i < I} \in P_{(-\infty,I)}(Y)$$

be such that

$$E_{-I} = \omega_+(Y, S_\Sigma)((\sigma_i)_{-\infty < i < I}),$$

and let

$$(\sigma_i)_{I \leqq i < \infty} \in E_I.$$

Then $(\sigma_i)_{i \in \mathbf{Z}} \in Y$, and

$$\omega_+(Y, S_\Sigma)(x_{(-\infty,i]}) = E_i, \qquad -I \leqq i \leqq I. \qquad \text{Q.e.d.}$$

Under the hypothesis, that $(Y, S_\Sigma)$ is topologically transitive with periodic points dense, more can be said about the projection $\rho_+(Y, S_\Sigma)$. As we shall see, $\rho_+(Y, S_\Sigma)$ is then one-to-one on a dense $G_\delta$, and at the same time we shall find that there is then in $\Omega_+(Y, S_\Sigma)$ a unique minimal ergodic class under $A_+(Y, S_\Sigma)$. For this we make some preparations. We say that a block $a \in P_{[j,k]}(Y)$, $j$, $k \in \mathbf{Z}$, $j < k$, is a finitary block if $\omega_+(Y, S_\Sigma)$ is constant on the set

$$\{x_- \in P_{(-\infty,k]}(Y) : P_{[j,k]}(x_-) = a\}.$$

The set of finitary blocks in $P_{[j,k]}(Y)$ will be denoted by $\mathscr{F}_{[j,k]}(Y, S_\Sigma)$. It is

$$S_\Sigma^i \mathscr{F}_{[j,k]}(Y, S_\Sigma) = \mathscr{F}_{[j-i,k-i]}(Y, S_\Sigma), \qquad i \in \mathbf{Z}.$$

Also every block that contains a finitary block as a subblock is itself finitary.

(2.4) LEMMA. *There exist finitary blocks.*

PROOF. Let

$$x_- = (x_i)_{-\infty < i \leqq 0} \in P_{(-\infty,0]}(Y)$$

be such that $\hat{\Delta}(x_-)$ contains a minimal number of elements. Denote for $k \in \mathbf{N}$ by $\hat{\Delta}_k$ the set of all $\hat{\sigma}_0 \in \hat{\Sigma}$ such that there is an $\hat{A}$-admissible block $(\hat{\sigma}_i)_{-k \leqq i \leqq 0}$ such that

$$x_i = \Phi(\hat{\sigma}_i), \qquad -k \leqq i \leqq 0.$$

By a compactness argument one has

$$\hat{\Delta}(x_-) = \bigcap_{k \in \mathbf{N}} \hat{\Delta}_k.$$

Hence there is a $k_0 \in \mathbf{N}$ such that

$$\hat{\Delta}(x_-) = \Delta_{k_0}.$$

We claim that $(x_i)_{-k_0 \leqq i \leqq 0}$ is a finitary block. Indeed, for all $x'_- \in P_{(-\infty,0]}(Y)$ such that

$$P_{[-k_0,0]}(x'_-) = (x_i)_{-k_0 \leqq i \leqq 0},$$

one has

$$\hat{\Delta}(x'_-) \subset \hat{\Delta}_{k_0}.$$

From this, since $\hat{\Delta}(x_-)$ is assumed to contain a minimal number of elements, one has

$$\hat{\Delta}(x'_-) = \hat{\Delta}(x_-),$$

and therefore from (1)

$$\omega_+(Y, S_\Sigma)(x'_-) = \omega_+(Y, S_\Sigma)(x_-). \qquad \text{Q.e.d.}$$

We say that a point $x \in Y$ is $F$-finitary if for all $i \in \mathbf{Z}$ there is a $j < i$ such that $x_{[j,i]}$ is a finitary block. We denote the set of $F$-finitary points by $F_+(Y, S_\Sigma)$. It is a $G_\delta$.

(2.5) LEMMA. *Let $(Y, S_\Sigma)$ be topologically transitive with periodic points dense. Then $F_+(Y, S_\Sigma)$ is dense in $Y$.*

PROOF. Let $I \in \mathbf{N}$, $a \in P_{[-I,I]}(Y)$, and let $f$ be any finitary block of $(Y, S_\Sigma)$. Use the topological transitivity of $(Y, S_\Sigma)$ to find $j < -I$ and $k > I$ and a block $b \in P_{[j,k]}(Y)$, that contains $f$ as a subblock, and such that

$$b_{[-I,I]} = a.$$

Every periodic point in $Z(b)$ is an $F$-finitary point in $Z(a)$. Q.e.d

For a finitary block $f \in \mathscr{F}_{[j,k)}(Y, S_\Sigma)$, $j$, $k \in \mathbf{Z}$, $j < k$, we set

$$\omega_+(Y, S_\Sigma)(f) = \omega_+(Y, S_\Sigma)(x_-), \qquad x_- \in Z(f) \cap P_{(-\infty,k)}(Y).$$

(2.6) LEMMA.

$$|\rho_+(Y, S_\Sigma)^{-1}\{x\}| = 1, \qquad x \in F_+(Y, S_\Sigma).$$

PROOF. Let

$$(x_i, D_i)_{i \in \mathbf{Z}} \in X_+(Y, S_\Sigma),$$

where

$$x \in F_+(Y, S_\Sigma).$$

Let $j$, $k \in \mathbf{Z}$, $j < k$, be such that $x_{[j,k)}$ is a finitary block. Then necessarily

$$D_k = S_\Sigma^{k-1} \omega_+(Y, S_\Sigma)(x_{[j,k)}). \qquad \text{Q.e.d.}$$

We denote by $\Xi_+^0(Y, S_\Sigma)$ the set of all sets $D$ in $\Xi_+(Y, S_\Sigma)$ such that for some finitary block $f \in \mathcal{F}_{[-I,0]}(Y, S_\Sigma)$, $I \in \mathbf{N}$,

$$D = \omega_+(Y, S_\Sigma)(f),$$

and we set

$$\Omega_+^0(Y, S_\Sigma) = \{(\sigma, D) \in \Omega_+(Y, S_\Sigma) : D \in \Xi_+^0(Y, S_\Sigma)\}.$$

(2.7) LEMMA. *Let $(Y, S_\Sigma)$ be topologically transitive with periodic points dense. Then $\Omega_+^0(Y, S_\Sigma)$ is the unique minimal ergodic class in $\Omega_+(Y, S_\Sigma)$ under $A_+(Y, S_\Sigma)$.*

PROOF. We have to show that for all $(\sigma_1, D) \in \Omega_+(Y, S_\Sigma)$ there is for some $I \in \mathbf{N}$ a finitary block $(\sigma_i)_{1 \leq i \leq I}$ such that

$$Z((\sigma_i)_{1 \leq i \leq I}) \cap D \neq \emptyset.$$

To see this, let $x_- \in P_{(-\infty,0]}(Y)$ be such that

$$D = \omega_+(Y, S_\Sigma)(x_-).$$

Let $f$ be any finitary block. Let then $k \in \mathbf{N}$, and use the topological transitivity and the density of the periodic points to find $I(k) \in \mathbf{N}$ and a block $a^{(k)} \in P_{[-k, I(k)]}(Y)$ such that

$$P_{[-k,0]}(a^{(k)}) = P_{[-k,0]}(x_-)$$

and such that $P_{[1,I(k)]}(a^{(k)})$ contains $f$ as a subblock. Recalling that $(Y, S_\Sigma)$ is the homomorphic image of $(X_{\tilde{A}}, S_\Sigma)$ one sees that one can have here $I(k)$ independent of $k$. A compactness argument yields then the lemma. Q.e.d.

The restriction $A_+^0(Y, S_\Sigma)$ of $A_+(Y, S_\Sigma)$ to the set $\Omega_+^0(Y, S_\Sigma)$ produces an irreducible topological Markov chain that we denote by $(X_+^0(Y, S_\Sigma), S_+^0(Y, S_\Sigma))$, and that we call the future finitary state chain of $(Y, S_\Sigma)$.

(2.8) THEOREM. *Let $(Y, S_\Sigma)$ be topologically transitive with periodic points dense. Then*

$$\rho_+(Y, S_\Sigma)(X_+^0(Y, S_\Sigma)) = Y.$$

PROOF. Let

$$a \in P_{[-I,I]}(Y), \qquad I \in \mathbf{N},$$

and let $f$ be a finitary block. Use the topological transitivity and the density of the periodic points to find an $x \in Y$ such that $x_{(-\infty,I)}$ contains $f$ as a subblock and such that

$$x_{[-I,I]} = a.$$

Then

$$(x_i, S_\Sigma^{i-1}\omega_+(Y, S_\Sigma)(x_{(-\infty,i)}))_{-I \leqq i \leqq I}$$

is an admissible block for $A_+^0(Y, S_\Sigma)$. It follows that $\rho_+(Y, S_\Sigma)(X_+^0(Y, S_\Sigma))$ is dense in $Y$ and the theorem is proved. Q.e.d.

We denote the restriction of $\rho_+(Y, S_\Sigma)$ to $X_+^0(Y, S_\Sigma)$ by $\rho_+^0(Y, S_\Sigma)$.

(2.9) COROLLARY. *Let $(Y, S_\Sigma)$ be topologically transitive with periodic points dense. Then every element of $\Xi_+(Y, S_\Sigma)$ contains an element of $\Xi_+^0(Y, S_\Sigma)$.*

PROOF. Let $D \in \Xi_+(Y, S_\Sigma)$, and for some

$$(x_i)_{-\infty < i \leqq 1} \in P_{(-\infty,1]}(Y)$$

let

$$D = \omega_+(Y, S_\Sigma)((x_i)_{-\infty < i \leqq 0}).$$

By Theorem (2.8) there is then a

$$(x_i, D_i^0)_{-\infty < i \leqq 0} \in P_{(-\infty,0]}(X_+^0(Y, S_\Sigma)).$$

If now

$$y_+ \in S_\Sigma P_{[1,\infty)}(Z(x_0) \cap D_0^0),$$

then by Lemma (2.2) also $y_+ \in D$. Q.e.d.

Reversing the direction of time in the constructions that we have carried out so far leads to analogue objects, where the place of the "future" is now taken by the "past". For these objects we use similar notation with a minus sign appearing instead of the plus sign. The topological Markov chain $(X_-(Y, S_\Sigma), S_-(Y, S_\Sigma))$ we call the past state chain of $(Y, S_\Sigma)$ and the topological Markov chain $(X_-^0(Y, S_\Sigma), S_-^0(Y, S_\Sigma))$ we call the past finitary state chain of $(Y, S_\Sigma)$.

Taking the fiber product of $(X_-(Y, S_\Sigma), S_-(Y, S_\Sigma))$ and $(X_+(Y, S_\Sigma), S_+(Y, S_\Sigma))$ with respect to the homomorphisms

$$\rho_-(Y, S_\Sigma): (X_-(Y, S_\Sigma), S_-(Y, S_\Sigma)) \to (Y, S_\Sigma),$$

$$\rho_+(Y, S_\Sigma): (X_+(Y, S_\Sigma), S_+(Y, S_\Sigma)) \to (Y, S_\Sigma),$$

produces a topological Markov chain, that we call the joint state chain of $(Y, S_\Sigma)$ and that we denote by $(X_{-+}(Y, S_\Sigma),\ S_{-+}(Y, S_\Sigma))$,

$$X_{-+}(Y, S_\Sigma) = \{(x_i, D_i^-, D_i^+)_{i\in\mathbf{Z}} \in \Sigma^{\mathbf{Z}} \times \Xi_-(Y, S_\Sigma) \times \Xi_+(Y, S_\Sigma):$$
$$(x_i, D_i^-)_{i\in\mathbf{Z}} \in X_-(Y, S_\Sigma),\ (x_i, D_i^+)_{i\in\mathbf{Z}} \in X_+(Y, S_\Sigma)\}.$$

Also, taking the fiber product of $(X_-^0(Y, S_\Sigma),\ S_-^0(Y, S_\Sigma))$ and $(X_+^0(Y, S_\Sigma),\ S_+^0(Y, S_\Sigma))$ with respect to the homomorphisms

$$\rho_0^-(Y, S_\Sigma) : (X_-^0(Y, S_\Sigma),\ S_-^0(Y, S_\Sigma)) \to (Y, S_\Sigma),$$
$$\rho_+^0(Y, S_\Sigma) : (X_+^0(Y, S_\Sigma),\ S_+^0(Y, S_\Sigma)) \to (Y, S_\Sigma),$$

produces a topological Markov chain, that we call the joint finitary state chain of $(Y, S_\Sigma)$ and that we denote by $(X_{-+}^0(Y, S_\Sigma),\ S_{-+}^0(Y, S_\Sigma))$,

$$X_{-+}^0(Y, S_\Sigma) = \{(x_i, D_i^-, D_i^+)_{i\in\mathbf{Z}} \in \Sigma^{\mathbf{Z}} \times \Xi_-^0(Y, S_\Sigma) \times \Xi_+^0(Y, S_\Sigma):$$
$$(x_i, D_i^-)_{i\in\mathbf{Z}} \in X_-^0(Y, S_\Sigma),\ (x_i, D_i^+)_{i\in\mathbf{Z}} \in X_+^0(Y, S_\Sigma)).$$

The properties of the projections

$$\pi_+(Y, S_\Sigma) : (X_{-+}(Y, S_\Sigma),\ S_{-+}(Y, S_\Sigma)) \to (X_+(Y, S_\Sigma), S_+(Y, S_\Sigma)),$$
$$\pi_+^0(Y, S_\Sigma) : (X_{-+}^0(Y, S_\Sigma),\ S_{-+}^0(Y, S_\Sigma)) \to (X_+^0(Y, S_\Sigma),\ S_+^0(Y, S_\Sigma)),$$
$$\pi_-(Y, S_\Sigma) : (X_{-+}(Y, S_\Sigma),\ S_{-+}(Y, S_\Sigma)) \to (X_-(Y, S_\Sigma),\ S_-(Y, S_\Sigma)),$$
$$\pi_-^0(Y, S_\Sigma) : (X_{-+}^0(Y, S_\Sigma),\ S_{-+}^0(Y, S_\Sigma)) \to (X_-^0(Y, S_\Sigma),\ S_-^0(Y, S_\Sigma)),$$

can be read off from the properties of the projections $\rho_-(Y, S_\Sigma)$, $\rho_-^0(Y, S_\Sigma)$, $\rho_+(Y, S_\Sigma)$, $\rho_+^0(Y, S_\Sigma)$. The left (right) resolving property of $\rho_-(Y, S_\Sigma)$ and $\rho_-^0(Y, S_\Sigma)(\rho_+(Y, S_\Sigma)$, and $\rho_+^0(Y, S_\Sigma))$ implies the left (right) resolving property of $\pi_+(Y, S_\Sigma)$ and $\pi_+^0(Y, S_\Sigma)(\pi_-(Y, S_\Sigma)$ and $\pi_+^0(Y, S_\Sigma))$, $\pi_-(Y, S_\Sigma)$ and $\pi_+(Y, S_\Sigma)$ possess Borel sections $\sigma_-(Y, S_\Sigma)$ and $\sigma_+(Y, S_\Sigma)$ that are given by

$$\sigma_-(Y, S_\Sigma)((x_i, D_i^-)_{i\in\mathbf{Z}}) = (x_i, D_i^-, D_i^+)_{i\in\mathbf{Z}},$$
$$(x_i, D_i^+)_{i\in\mathbf{Z}} = \tau_+(Y, S_\Sigma)((x_i)_{i\in\mathbf{Z}}),$$
$$(x_i, D_i^-)_{i\in\mathbf{Z}} \in X_-(Y, S_\Sigma)$$

$$\sigma_+(Y, S_\Sigma)(x_i, D_i^+)_{i\in\mathbf{Z}}) = (x_i, D_i^-, D_i^+)_{i\in\mathbf{Z}},$$
$$(x_i, D_i^-)_{i\in\mathbf{Z}} = \tau_-(Y, S_\Sigma)((x_i)_{i\in\mathbf{Z}}),$$
$$(x_i, D_i^+)_{i\in\mathbf{Z}} \in X_+(Y, S_\Sigma).$$

$\pi_-(Y, S_\Sigma)$ is one-to-one on the set

$$\{(x_i, D_i^-, D_i^+)_{i\in\mathbf{Z}} \in X_{-+}(Y, S_\Sigma) : (x_i)_{i\in\mathbf{Z}} \in F_+(Y, S_\Sigma)\},$$

and $\pi_+(Y, S_\Sigma)$ is one-to-one on the set

$$\{(x_i, D_i^-, D_i^+)_{i\in\mathbf{Z}} X_{-+}(Y, S_\Sigma) : (x_i)_{i\in\mathbf{Z}} \in F_-(Y, S_\Sigma)\}.$$

We also remark that the closure of the set

$$\sigma_-(Y, S_\Sigma)\tau_+(Y, S_\Sigma)(F_-(Y, S_\Sigma) \cap F_+(Y, S_\Sigma))$$
$$= \sigma_+(Y, S_\Sigma)\tau_-(Y, S_\Sigma)(F_-(Y, S_\Sigma) \cap F_+(Y, S_\Sigma))$$

is a basic set of $(X^0_{-+}(Y, S_\Sigma), S^0_{-+}(Y, S_\Sigma))$.

In the remainder of this section we consider two topologically conjugate sofic systems $(Y, S_\Sigma)$ and $(\bar{Y}, S_{\bar{\Sigma}})$. We let

$$u : (Y, S_\Sigma) \to (\bar{Y}, S_{\bar{\Sigma}})$$

be a topological conjugacy that together with its inverse is implemented by $(2N+1)$-block maps $\psi$ and $\bar{\psi}$,

$$\psi : P_{[-N,N]}(Y) \to \bar{\Sigma},$$
$$\bar{\psi} : P_{[-N,N]}(\bar{Y}) \to \Sigma.$$

Thus

(3) $$ux = (\psi(x_{[i-N,i+N]}))_{i\in\mathbf{Z}}, \qquad (x_i)_{i\in\mathbf{Z}} \in Y.$$

(4) $$\bar{u}\bar{x} = (\bar{\psi}(\bar{x}_{[i-N,i+N]}))_{i\in\mathbf{Z}}, \qquad (\bar{x}_i)_{i\in\mathbf{Z}} \in Y.$$

(2.10) LEMMA.

$$uF_+(Y, S_\Sigma) = F_+(\bar{Y}, S_{\bar{\Sigma}}).$$

PROOF. Let $x \in F_+(Y, S_\Sigma)$. We show that $\bar{x} = ux$ is in $F_+(\bar{Y}, S_{\bar{\Sigma}})$. Let $i \in \mathbf{Z}$ and let $x_{[j,i)}$ be a finitary block of $(Y, S_\Sigma)$, $i - j > N$. We want to show that $\bar{x}_{[j-N,i+N)}$ is then a finitary block of $(\bar{Y}, S_{\bar{\Sigma}})$. For this, let

$$\bar{x}'_-, \bar{x}''_- \in P_{(-\infty,i+N)}(Z(\bar{x}_{[j-N,i+N)}) \cap \bar{Y}).$$

We have to show that then

(5) $$\omega_+(\bar{Y}, S_{\bar{\Sigma}})(\bar{x}'_-) = \omega_+(\bar{Y}, S_{\bar{\Sigma}})(\bar{x}''_-).$$

Let

$$\bar{y}_+ \in \omega_+(\bar{Y}, S_{\bar{\Sigma}})(\bar{x}'_-),$$

and set

$$y_+ = P_{[i,\infty)}(u^{-1}(\bar{x}'_-, \bar{y}_+)),$$

and also let $x'_-$, $x''_- \in P_{(-\infty,i)}(Y)$ be given by

$$x'_- = (\bar{\psi}(P_{[k-N,k+N]}(\bar{x}'_-))_{-\infty<k\leq i},$$

$$x''_- = (\bar{\psi}(P_{[k-N,k+N]}(\bar{x}''_-))_{-\infty<k\leq i}.$$

Since $i-j>N$, by (4)

$$y_+ \in \omega_+(Y, S_\Sigma)(x'_-)$$

and also

(6) $$x'_-, x''_- \in Z(x_{[j,i)}),$$

and, since $x_{[j,i)}$ is finitary for $(Y, S_\Sigma)$,

$$y_+ \in \omega_+(Y, S_\Sigma)(x''_-).$$

Therefore $(x''_-, y_+) \in Y$, and, again since $i-j>N$, one has by (3) and (6) that

$$u(x''_-, y_+) = (\bar{x}''_-, \bar{y}_+).$$

Thus

$$\bar{y}_+ \in \omega_+(\bar{Y}, S_{\bar{\Sigma}})(\bar{x}''_-),$$

and (5) follows. Q.e.d.

Our next task is to construct a topological conjugacy

$$u_+ : (X_+(Y, S_\Sigma),\ S_+(Y, S_\Sigma)) \to (X_+(\bar{Y}, S_{\bar{\Sigma}}),\ S_+(\bar{Y}, S_{\bar{\Sigma}})).$$

To describe this conjugacy, let

$$x \in Y, \qquad (x_i, D_i)_{i\in\mathbf{Z}} \in X_+(Y, S_\Sigma),$$

set $\bar{x} = ux$, and

$$E_i = S_\Sigma^{-i+1} D_i, \qquad i \in \mathbf{Z}.$$

By Lemma (2.2) we can set

$$E'_{i-N} = \{y_+ \in E_{i-N} : P_{(-\infty,i)}(u(x_{(-\infty,i-N)}, y_+)) = \bar{x}_{(-\infty,i)}\},$$

$$\bar{E}_i = \{P_{[i,\infty)}(u(x_{(-\infty,i-N)}, y_+)) : y_+ \in E'_{i-N}\}.$$

(2.11) LEMMA. *Let* $i \in \mathbf{Z}$,

$$M > 3N, \tag{7}$$

*and let*

$$x'_- \in P_{(-\infty, i-M)}(Y)$$

*be such that*

$$E_{i-M} = \omega_+(Y, S_\Sigma)(x'_-). \tag{8}$$

*Set*

$$\bar{x}'_- = P_{(-\infty, i-M+N)}(u(x'_-, x_{[i+M,\infty)})). \tag{9}$$

*Then*

$$\bar{E}_i = \omega_+(\bar{Y}, S_{\bar{\Sigma}})(\bar{x}'_-, \bar{x}_{[i-M+N,i)}).$$

PROOF. For the proof, let first $\bar{y}_+ \in \bar{E}_i$. This means that there is a $y_+ \in E_{i-N}$ such that

$$P_{[i,\infty)}(u(x_{(-\infty,i-N)}, y_+)) = \bar{y}_+, \tag{10}$$

and

$$P_{[i-2N,i)}(u(x_{(-\infty,i-N)}, y_+)) = \bar{x}_{[i-2N,i)}. \tag{11}$$

Using (3) one has from (10) that

$$P_{[i,\infty)}(u(x'_-, x_{[i-M,i-N)}, y_+)) = \bar{y}_+$$

and using (3) and (7) one has from (9) and (11) that

$$P_{(-\infty,i)}(u(x'_-, x_{[i-M,i-N)}, y_+)) = (\bar{x}'_-, \bar{x}_{[i-M+N,i)}),$$

and one sees that

$$\bar{y}_+ \in \omega_+(\bar{Y}, S_{\bar{\Sigma}})(\bar{x}_-, \bar{x}_{[i-M+N,i)}). \tag{12}$$

On the other hand, if (12) is assumed, set

$$y_+ = P_{[i-N,\infty)}(u^{-1}(\bar{x}'_-, \bar{x}_{[i-M+N,i)}, \bar{y}_+), \tag{13}$$

and have from (4) and (9) that

$$P_{(-\infty,i-N)}(u^{-1}(\bar{x}'_-, \bar{x}_{[i-M+N,i)}, \bar{y}_+)) = (x'_-, x_{[i-M,i-N)}).$$

Hence, by (8),

$$y_+ \in E_{i-N}.$$

Also by (13) and (12)

$$P_{[i,\infty)}(u(x_{(-\infty,i-N)},y_+))=\bar{y}_+$$

and

$$P_{[i-2N,i)}(u(x_{(-\infty,i-N)},y_+))=\bar{x}_{[i-2N,i)}.$$

Therefore

$$\bar{y}_+\in\bar{E}_i. \qquad \text{Q.e.d.}$$

(2.12) Lemma.

$$\bar{E}_k=P_{[k,\infty)}(Z(\bar{x}_{[j,k]})\cap\bar{E}_j),\qquad j,k\in\mathbf{Z},\quad i<k.$$

Proof. Use Lemma (2.11). Q.e.d.

$\bar{E}_i$ is determined by $E_{i-N}$ and by $x_{[i-2N,i-N]}$. Therefore, as is seen from Lemma (2.12), by setting

$$\bar{D}_i=S_{\bar{\Sigma}}^{i-1}\bar{E}_i,\qquad i\in\mathbf{Z}$$

we assign to the point $(x_i,D_i)_{i\in\mathbf{Z}}$ in a continuous and shift-invariant manner a point

$$u_+((x_i,D_i)_{i\in\mathbf{Z}})=(\bar{x}_i,\bar{D}_i)_{i\in\mathbf{Z}}\in X_+(\bar{Y},S_{\bar{\Sigma}}).$$

(2.13) Theorem. *$u_+$ is a topological conjugacy of $(X_+(Y,S_\Sigma),\ S_+(Y,S_\Sigma))$ onto $(X_+(\bar{Y},S_{\bar{\Sigma}}),\ S_+(\bar{Y},S_{\bar{\Sigma}}))$.*

Proof. Set

$$\bar{E}'_i=\{\bar{y}_+\in\bar{E}_i:P_{(-\infty,i+N)}(u^{-1}(\bar{x}_{(-\infty,i)},\bar{y}_+))=x_{(-\infty,i+N)}\},$$

$$\bar{\bar{E}}_{i+N}=\{P_{[i+N,\infty)}(u^{-1}(\bar{x}_{(-\infty,i)},\bar{y}_+)):\bar{y}_+\in\bar{E}'_i\},\ i\in\mathbf{Z}.$$

We show that

$$\bar{\bar{E}}_{i+N}=E_{i+N},\qquad i\in\mathbf{Z}.$$

To see this, let first

$$y_+\in\bar{\bar{E}}_{i+N}.$$

This means that there is a

$$\bar{y}_+\in\bar{E}'_i$$

such that

$$y_+ = P_{[i+N,\infty)}(u^{-1}(\bar{x}_{(-\infty,i)}, \bar{y}_+))$$

and

$$P_{[i-N,i+N)}(u^{-1}(\bar{x}_{(-\infty,i)}, \bar{y}_+)) = x_{[i-N,i+N)}.$$

Then

(14) $$u^{-1}(\bar{x}_{(-\infty,i)}, \bar{y}_+) = (x_{(-\infty,i+N)}, y_+).$$

Further there is a

$$\tilde{y}_+ \in E_{i-N}$$

such that

$$\bar{y}_+ = P_{[i,\infty)}(u(x_{(-\infty,i-N)}, \tilde{y}_+))$$

and

$$P_{[i-2N,i)}(u(x_{(-\infty,i-N)}, \tilde{y}_+)) = \bar{x}_{[i-2N,i)}.$$

Since $u$ is one-to-one it follows from this and from (14) that

$$P_{[i-N,i+N)}\tilde{y}_+ = x_{[i-N,i+N)}, \qquad P_{[i+N,\infty)}\tilde{y}_+ = y_+,$$

and this means that

$$y_+ \in P_{[i+N,\infty)}(Z(x_{(i-N,i+N)}) \cap E_{i-N}) = E_{i+N}.$$

On the other hand, if

$$y_+ \in E_{i+N}$$

then

$$(x_{[i-N,i+N)}, y_+) \in E'_{i-N}.$$

Setting

$$\bar{y}_+ = P_{[i,\infty)}u(x_{(-\infty,i+N)}, y_+),$$

one has then again (14) and it follows that

$$\bar{y}_+ \in \bar{E}_i, \qquad y_+ \in \bar{\bar{E}}_{i+N}.$$

Interchanging $(Y, S_\Sigma)$ and $(\bar{Y}, S_{\bar{\Sigma}})$ and repeating the construction one sees that $u_+$ is a topological conjugacy as claimed. Q.e.d.

(2.14) THEOREM. *$u_+$ is the unique topological conjugacy of $(X_+(Y, S_\Sigma), S_+(Y, S_\Sigma))$ onto $(X_+(\bar{Y}, S_{\bar{\Sigma}}), S_+(\bar{Y}, S_{\bar{\Sigma}}))$ such that*

$$u\rho_+(Y, S_\Sigma) = \rho_+(\bar{Y}, S_{\bar{\Sigma}})u_+,$$

$$u_+\tau_+(Y, S_\Sigma) = \tau_+(\bar{Y}, S_{\bar{\Sigma}})u.$$

PROOF. Let $x \in Y$, $\bar{x} = ux$. That

$$u_+\tau_+(Y, S_\Sigma)(x) = u_+((x_i, \omega_+(Y, S_\Sigma)(x_{(-\infty,i)}))_{i\in\mathbf{Z}})$$
$$= ((\bar{x}_i, \omega_+(\bar{Y}, S_{\bar{\Sigma}})(\bar{x}_{(-\infty,i]}))_{i\in\mathbf{Z}}) = \tau_+(\bar{Y}, S_{\bar{\Sigma}})(\bar{x}),$$

follows from Lemma (2.11). The uniqueness statement is a consequence of Lemma (2.3). Q.e.d.

(2.15) LEMMA. *Let $(Y, S_\Sigma)$ and $(Y, S_{\bar{\Sigma}})$ be topologically transitive with periodic points dense. Then*

$$u_+X^0_+(Y, S_\Sigma) = X^0_+(\bar{Y}, S_{\bar{\Sigma}}).$$

PROOF. Use Lemma (2.7) and Theorem (2.13). Q.e.d.

For topologically transitive sofic systems $(Y, S_\Sigma)$ and $(\bar{Y}, S_{\bar{\Sigma}})$ with periodic points dense we restrict $u_+$ to $X^0_+(Y, S_\Sigma)$ to obtain a topological conjugacy $u^0_+$ of $(X^0_+(Y, S_\Sigma), S^0_+(Y, S_\Sigma))$ onto $(X^0_+(Y, S_{\bar{\Sigma}}), S^0_+(\bar{Y}, S_{\bar{\Sigma}}))$.

(2.16) COROLLARY. *Let $(Y, S_\Sigma)$ and $(\bar{Y}, S_{\bar{\Sigma}})$ be topologically transitive with periodic points dense. Then $u^0_+$ is the unique topological conjugacy of $(X^0_+(Y, S_\Sigma), S^0_+(Y, S_S))$ onto $(X^0_+(\bar{Y}, S_{\bar{\Sigma}}), S^0_+(\bar{Y}, S_{\bar{\Sigma}}))$ such that*

$$u\rho^0_+(Y, S_\Sigma) = \rho^0_+(\bar{Y}, S_{\bar{\Sigma}})u_+.$$

PROOF. Apply Theorem (2.15) in conjunction with Lemmas (2.5), (2.6) and (2.10). Q.e.d.

Reversing the direction of time one produces by the same construction a topological conjugacy

$$u_-: (X_-(Y, S_\Sigma), S_-(Y, S_\Sigma)) \to (X_-(\bar{Y}, S_{\bar{\Sigma}}), S_-(\bar{Y}, S_{\bar{\Sigma}})),$$

and its restriction

$$u^0_-: (X^0_-(Y, S_\Sigma), S^0_-(Y, S_\Sigma)) \to (X^0_-(\bar{Y}, S_{\bar{\Sigma}}), S^0_-(\bar{Y}, S_{\bar{\Sigma}})).$$

One has then also a topological conjugacy

$$u_{-+}: (X_{-+}(Y, S_\Sigma), S_{-+}(Y, S_\Sigma)) \to (X_{-+}(\bar{Y}, S_{\bar{\Sigma}}), S_{-+}(\bar{Y}, S_{\bar{\Sigma}})).$$

Here $u_{-+}$ carries a point

$$(x_i, D_i^-, D_i^+)_{i\in\mathbf{Z}} \in X_{-+}(Y, S_\Sigma)$$

into the point

$$(\bar{x}_i, \bar{D}_i^-, \bar{D}_i^+)_{i\in\mathbf{Z}} \in X_{-+}(\bar{Y}, S_{\bar{\Sigma}})$$

that is given by

$$(\bar{x}_i, \bar{D}_i^-)_{i\in\mathbf{Z}} = u_-((x_i, D_i^-)_{i\in\mathbf{Z}}),$$

$$(\bar{x}_i, \bar{D}_i^+)_{i\in\mathbf{Z}} = u_+((x_i, D_i^+)_{i\in\mathbf{Z}}).$$

By restricting $u_{-+}$ one has finally a topological conjugacy

$$u_{-+}^0 : (X_{-+}^0(Y, S_\Sigma),\ S_{-+}^0(Y, S_\Sigma)) \to (X_{-+}^0(\bar{Y}, S_{\bar{\Sigma}}),\ S_{-+}^0(\bar{Y}, S_{\bar{\Sigma}})).$$

Uniqueness statements analogous to the ones in Theorem (2.15) and Corollary (2.16) hold for $u_{-+}$ and $u_{-+}^0$.

## 3. Dimension

Consider again a finite state space $\Sigma$. For an $S_\Sigma$-invariant set $Y \subset \Sigma^{\mathbf{Z}}$, and for a set $H \subset Y$ we denote by $W_Y^-(H)$ ($W_Y^+(H)$) the set of all points in $Y$ that are negatively (positively) asymptotic to a point in $H$. For a zero-one transition matrix $(A(\sigma,\sigma'))_{\sigma,\sigma'\in\Sigma}$, we denote

$$\Sigma[A] = \{\sigma \in \Sigma : |\{(\sigma_i)_{-\infty<i\leq 0} \in P_{(-\infty,0]}(X_A) : \sigma_0 = \sigma\}| < \infty\},$$

and

$$R_\sigma[A] = |\{(\sigma_i)_{-\infty<i\leq 0} \in P_{(-\infty,0]}(X_A) : \sigma_0 = \sigma\}|, \quad \sigma \in \Sigma[A].$$

(3.1) Lemma. *Let $H \subset X_A$ be $S_\Sigma$-invariant and such that*

(15) $$W_{X_A}^+(W_{X_A}^-(H)) = X_A.$$

*Then*

(16) $$|\{(\sigma_i)_{-\infty<i\leq 0} \in P_{(-\infty,0]}(W_{X_A}^-(H)) : \sigma_0 = \sigma\}| = R_\sigma[A], \qquad \sigma \in \Sigma[A],$$

*and if $\sigma \in \Sigma - \Sigma[A]$, then*

$$\{(\sigma_i)_{-\infty<i\leq 0} \in P_{(-\infty,0]}(W_{X_A}^-(H)) : \sigma_0 = \sigma\}$$

*is an infinite set.*

PROOF. Denote by $Q$ the set of all points $x$ in $X_A$, necessarily periodic, with the property that every symbol that appears in $x$ has a unique predecessor under $A$. If for $\sigma \in \Sigma[A]$,

$$(\sigma_i)_{-\infty<i\leqq 0} \in P_{(-\infty,0]}(X_A), \qquad \sigma_0 = \sigma,$$

then

$$(\sigma_i)_{-\infty<i\leqq 0} \in P_{(-\infty,0]} W^-_{X_A}(Q).$$

(15) implies that $H$ contains for every $x \in Q$ an element that is negatively asymptotic to $x$. This proves (16).

On the other hand, if $\sigma \in \Sigma - \Sigma[A]$, then one has two periodic points $y_1$, $y_2 \in X_A$, such that

$$y_2 \in W^-_{X_A}(\{y_1\}),$$

and such that there is a

$$(\sigma_i)_{-\infty<i\leqq 0} \in P_{(-\infty,0]}(X_A), \qquad \sigma_0 = \sigma,$$

that is negatively asymptotic to $y_2$. By (15)

$$y_1 \in W^-_{X_A}(H),$$

and one constructs an at least countable infinity of points

$$(\sigma_i)_{-\infty<i\leqq 0} \in P_{(-\infty,0]}(W^-_{X_A}(H)), \qquad \sigma_0 = \sigma. \qquad \text{Q.e.d.}$$

Condider now again a sofic system $(Y, S_\Sigma)$, and let $H \subset Y$ be a finite or countably infinite $S_\Sigma$-invariant set. Write

$$W^-_Y(H) = \bigcup_{n\in\mathbf{N}} \bigcup_{(h_i)_{i\in\mathbf{Z}}\in H} \{y \in Y : y_i = h_i, i \leqq -n\},$$

and put on $W^-_Y(H)$ a topology that turns it into a $\sigma$-compact space, by using on the sets

$$\{y \in Y : y_i = h_i, i \leqq -n\}, \qquad n \in \mathbf{N}, \quad (h_i)_{i\in\mathbf{Z}} \in H$$

the compact topology that they inherit from $Y$, and by furnishing $W^-_Y(H)$ with the inductive limit topology.

One introduces next a group $G_H$ of homeomorphisms of $W^-_Y(H)$, that we call the group of uniformly finite dimensional homeomorphisms of $W^-_Y(H)$. Here a homeomorphism $v$ of $W^-_Y(H)$ is in $G_H$ if and only if there is an $M \in \mathbf{N}$ such that

$$P_{[M,\infty)}(vy) = y_{[M,\infty)}, \qquad y \in W^-_Y(H).$$

Consider next the Boolean ring $\mathscr{C}_H$ of compact open subsets of $W_Y^-(H)$. The group $G_H$ acts on $\mathscr{C}_H$ and we obtain a (future) dimension function $\delta_H$ which is the quotient map of $\mathscr{C}_H$ onto the orbit space of this action. The range of $\delta_H$ carries an algebraic structure, where for $\gamma, \gamma' \in \delta_H(\mathscr{C}_H)$,

$$\gamma + \gamma' = \delta_H(C \cup C'), \qquad C \in \gamma, \quad C' \in \gamma', \quad C \cap C' = \varnothing.$$

$H$ being $S_\Sigma$-invariant we have also $W_Y^-(H)$ $S_\Sigma$-invariant, and

$$S_\Sigma G_H S_\Sigma^{-1} = G_H.$$

We have therefore an automorphism $\varphi_H$ of $\delta_H(\mathscr{C}_H)$ induced by $S_\Sigma$.

To specify further the pair $(\delta_H(\mathscr{C}_H), \varphi_H)$ we make now a suitable choice of the set $H$. Say that an $S_\Sigma$-invariant set $H \subset Y$ is dimensionally covering for $(Y, S_\Sigma)$ if $H$ is finite or countably infinite and if for all $D \in \Xi_+(Y, S_\Sigma)$ there is an

$$x_- \in P_{(-\infty,0]}(W_Y^-(H))$$

such that

$$\omega_+(Y, S_\Sigma)(x_-) = D.$$

We want to show that $(\delta_H(\mathscr{C}_H), \varphi_H)$ does not depend on the choice of $H$, as long as $H$ is dimensionally covering. For this we have the following two lemmas. Here we set, for $(\sigma, D) \in \Omega_+(Y, S_\Sigma)$,

$$\mathscr{K}_H(\sigma, D) = \{(\sigma_i)_{-\infty \leq i \leq 1} \in P_{(-\infty,1]}(W_Y^-(H)) : \sigma = \sigma_1, D = \omega_+(Y, S_\Sigma)((\sigma_i)_{-\infty < i \leq 0})\}.$$

(3.2) Lemma. *Let $H$ be dimensionally covering for $(Y, S_\Sigma)$. Then*

$$|\mathscr{K}_H(\sigma, D)| = R_{(\sigma,D)}[A_+(Y, S_\Sigma)], \quad (\sigma, D) \in \Omega_+(Y, S_\Sigma)[A_+(Y, S_\Sigma)],$$

*and if*

$$(\sigma, D) \in \Omega_+(Y, S_\Sigma) - \Omega_+(Y, S_\Sigma)[A_+(Y, S_\Sigma)],$$

*then $\mathscr{K}_H(\sigma, D)$ is an infinite set.*

Proof. That $H$ is dimensionally convering for $(Y, S_\Sigma)$ means that $\tau_+(Y, S_\Sigma)(H)$ is $S_+(Y, S_\Sigma)$-invariant, and that

$$W^+_{X_+(Y,S_\Sigma)}(W^-_{X_+(Y,S_\Sigma)}(\tau_+(Y, S_\Sigma)(H))) = X_+(Y, S_\Sigma).$$

Also, observe that for all $(\sigma, D) \in \Omega_+(Y, S_\Sigma)$ the mapping

$$(\sigma_i)_{-\infty < i \leq 1} \to (\sigma_i, \omega_+(Y, S_\Sigma)((\sigma_j)_{-\infty < j < i}))_{-\infty < i \leq 1}, \quad ((\sigma_i)_{-\infty < i \leq 1} \in \mathscr{K}_H(\sigma, D)),$$

is one-to-one and onto the set

$$\{(\sigma_i, D_i)_{-\infty<i\leq 1} \in P_{(-\infty,1]}(W^{-}_{X_+(Y,S_\Sigma)}(\tau_+(Y,S_\Sigma)H)) : (\sigma_1, D_1) = (\sigma, D)\},$$

and the present lemma follows from Lemma (3.1). Q.e.d.

(3.3) LEMMA. *Let $H$ and $H'$ be dimensionally covering sets for $(Y, S_\Sigma)$. Then there exists a one-to-one and onto mapping*

$$w : W_Y(H) \to W_Y(H')$$

*such that*

$$P_{[1,\infty)}(w(x)) = x_{[1,\infty)}, \qquad x \in W^{-}_Y(H). \tag{17}$$

PROOF. By Lemma (3.2) we have for all $(\sigma, D) \in \Omega_+(Y, S_\Sigma)$ one-to-one and onto mappings

$$\pi(\sigma, D) : \mathcal{K}_H(\sigma, D) \to \mathcal{K}_H(\sigma, D).$$

Define the mapping $w$ by requiring that

$$w(Z(x_-)) = Z(\pi(\sigma, D)(x_-)), \qquad x_- \in \mathcal{K}_H(\sigma, D), \quad (\sigma, D) \in \Omega_+(Y, S_\Sigma),$$

and by stipulating (17). Q.e.d.

View the mapping $w$ of Lemma (3.3) as a one-to-one and onto mapping $w : \mathcal{C}_H \to \mathcal{C}_{H'}$. Then

$$w\mathcal{G}_H w^{-1} = \mathcal{G}_H,$$

$$\delta_H(w^{-1}S_\Sigma wC) = \delta_H(S_\Sigma C), \qquad C \in \mathcal{C}_H.$$

Therefore the pair $(\delta_H(C_H), \varphi_H)$ does not depend on the choice of the dimensionally covering set $H$, and we write for it $(\delta_{(Y,S_\Sigma)}(\mathcal{C}_{(Y,S_\Sigma)}), \varphi_{(Y,S_\Sigma)})$. The definition of this pair is in fact intrinsic. To see this, consider again two topologically conjugate sofic systems $(Y, S_\Sigma)$, $(\bar{Y}, S_{\bar{\Sigma}})$, and let

$$u : (Y, S_\Sigma) \to (\bar{Y}, S_{\bar{\Sigma}})$$

be a topological conjugacy. For a finite or countably infinite set $H \subset Y$ the definition of the topology on $W^{-}_Y(H)$ is intrinsic in the sense that $u$, when restricted to $W^{-}_Y(H)$, becomes a homeomorphism $u_H$ of $W^{-}_Y(H)$ onto $W^{-}_{\bar{Y}}(uH)$. The definition of the group of uniformly finite dimensional homeomorphisms is intrinsic in the sense that

$$u_H \mathcal{G}_H u_H^{-1} = \mathcal{G}_{uH}.$$

Finally the notion of a dimensionally covering set is intrinsic as is seen from the following lemma.

(3.4) LEMMA. *$H$ is dimensionally covering for $(Y, S_\Sigma)$ if and only if $uH$ is dimensionally covering for $(\bar{Y}, S_{\bar{\Sigma}})$.*

PROOF. A finite or countably infinite set $H \subset Y$ is dimensionally covering for $(Y, S_\Sigma)$ if and only if

$$W^+_{X_+(Y,S_\Sigma)}(W^-_{X_+(Y,S_\Sigma)}(\tau_+(Y, S_\Sigma)H)) = X_+(Y, S_\Sigma).$$

Apply Theorem (2.14). Q.e.d.

(3.5) THEOREM. *There is an isomorphism of pairs*

$$(\delta_{(Y,S_\Sigma)}(\mathscr{C}_{(Y,S_\Sigma)}), \varphi_{(Y,S_\Sigma)}) \to (\delta_{(X_+(Y,S_\Sigma),S_+(Y,S_\Sigma))}(\mathscr{C}_{(X_+(Y,S_\Sigma),S_+(Y,S_\Sigma))}), \varphi_{(X_+(Y,S_\Sigma),S_+(Y,S_\Sigma))}).$$

PROOF. Let $(\sigma, D) \in \Omega_+(Y, S_\Sigma)$. Consider a dimensionally covering set $H \subset Y$. Let $i \in \mathbf{Z}$.

$$x_-, x'_- \in S_\Sigma^{-i+1}\mathscr{K}_H(\sigma, D).$$

We claim that

$$\delta_{(Y,S_\Sigma)}(Z(x_-) \cap Y) = \delta_{(Y,S_\Sigma)}(Z(x'_-) \cap Y).$$

In fact, an element $v$ of $\mathscr{G}_H$ that carries $Z(x_-) \cap Y$ onto $Z(x'_-) \cap Y$ is defined by setting

$$v(Z(x_-) \cap Y) = Z(x'_-) \cap Y,$$

$$P_{(i,\infty)}(vy) = y_{(i,\infty)},$$

$$vy = y, \qquad y \in W_Y(H) - ((Z(x_-) \cup Z(x'_-)) \cap Y).$$

It is therefore meaningful to define elements $\delta_i(\sigma, D)$ of $\delta_{(Y,S_\Sigma)}(\mathscr{C}_{(Y,S_\Sigma)})$ by setting

$$\delta_i(\sigma, D) = \delta_{(Y,S_\Sigma)}(Z(x_-) \cap Y), \quad x \in S_\Sigma^{i+1}\mathscr{K}_H(\sigma, D), (\sigma, D) \in \Omega_+(Y, S_\Sigma), \ i \in \mathbf{Z}.$$

Every element of $\mathscr{C}_H$ is a finite disjoint union of sets of the form $Z(x_-) \cap Y$, $x_- \in P_{(-\infty,i)}(W_Y(H))$, $i \in \mathbf{Z}$. Therefore the $\delta_i(\sigma, D)$, $i \in \mathbf{Z}$, $(\sigma, D) \in \Omega_+(Y, S_\Sigma)$ span the dimension range. Moreover writing a $Z(x_-) \cap Y$, $x_- \in P_{(-\infty,i)}(W^-_Y(H))$ as a disjoint union of cylinder sets of the form $Z(y_-) \cap Y$, $y_- \in P_{(-\infty,i]}(W^-_Y(H))$ shows that

$$\delta_{i-1}(\sigma, D) = \sum_{(\sigma',D') \in \Omega_+(Y,S_\Sigma)} A_+((\sigma, D), (\sigma', D'))\delta_i(\sigma', D').$$

From this it is seen that $\delta_{(Y,S_\Sigma)}(\mathscr{C}_{(Y,S_\Sigma)})$ is a segment of an ordered abelian group, the (future) dimension group of $(Y,S_\Sigma)$, that is given by

$$\varinjlim_{A_+(Y,S_\Sigma)^{\mathrm{T}}} (\mathbf{Z}^{\Omega_+(Y,S_\Sigma)}, \mathbf{Z}_+^{\Omega_+(Y,S_\Sigma)}),$$

$\varphi_{(Y,S_\Sigma)}$ is given by the automorphism that $A_+(Y,S_\Sigma)^{\mathrm{T}}$ induces on this direct limit. The description of the pair

$$(\delta_{(X_+(Y,S_\Sigma),S_+(Y,S_\Sigma))}(\mathscr{C}_{(X_+(Y,S_\Sigma),S_+(Y,S_\Sigma))}),\ \varphi_{(X_+(Y,S_\Sigma),S_+(Y,S_\Sigma))})$$

is identical (compare [9]). Indeed, the isomorphism that carries one pair into the other is obtained by mapping $\delta_i(\sigma,D)$ into

$$\delta_{(X_+(Y,S_\Sigma),S_+(Y,S_\Sigma))}(\tau_+(Z(x_-)\cap Y)),$$

$$x_- \in S_\Sigma^{-i+1}\mathscr{K}(\sigma,D), \qquad (\sigma,D)\in\Omega_+(Y,S_\Sigma), \qquad i\in\mathbf{Z}. \qquad \text{Q.e.d.}$$

Note that, with the notation

$$\mathscr{R}(\sigma,D)=\begin{cases}\mathbf{Z}_+ & \text{if } (\sigma,D)\in\Omega_+(Y,S_\Sigma)-\Omega_+(Y,S_\Sigma)[A_+(Y,S_\Sigma)],\\ [0,R_{(\sigma,D)}[A_+(Y,S_\Sigma)]], & \text{if } (\sigma,D)\in\Omega_+(Y,S_\Sigma)[A_+(Y,S_\Sigma)],\end{cases}$$

the dimension range of $(Y,S_\Sigma)$ as such is given by

$$\varinjlim_{A_+(Y,S_\Sigma)^{\mathrm{T}}} \prod_{(\sigma,D)\in\Omega_+(Y,S_\Sigma)} \mathscr{R}(\sigma,D).$$

Also note for completeness that reversing the direction of time in the construction yields an analogue (past) dimension.

Given two topologically conjugate sofic systems $(Y,S_\Sigma)$ and $(\bar{Y},S_{\bar{\Sigma}})$ one has from Theorem (3.5) that $A_+(Y,S_\Sigma)$ and $A_+(\bar{Y},S_{\bar{\Sigma}})$ are shift equivalent (compare theorem (4.2) of [9]). The topological conjugacy of $(X_+(Y,S_\Sigma),\ S_+(Y,S_\Sigma))$ and $(X_+(\bar{Y},S_{\bar{\Sigma}}),\ S_+(\bar{Y},S_{\bar{\Sigma}}))$ is therefore a consequence of William's conjecture [14]. We have seen in Theorem (2.14) that this consequence holds.

## 4. Proper soficity

Consider again a finite state space $\Sigma$ and a sofic system $(Y,S_\Sigma)$. Denote by $C(Y,S_\Sigma)$ the set of all $x\in Y$ that possess a finitary subblock.

(4.1) PROPOSITION. *$(Y,S_\Sigma)$ is Markov if and only if $Y=C(Y,S_\Sigma)$.*

PROOF. If every point in $Y$ possesses an $F$-finitary subblock, then

$$Y = \bigcup_{j<0<k} \bigcup_{f \in \mathcal{F}_{[j,k]}(Y,S_\Sigma)} Z(f).$$

Hence, by compactness, we have an $N \in \mathbf{N}$ such that every block in $P_{[1,N]}(Y)$ is $F$-finitary. One defines then a topological Markov chain with state space $P_{[1,N]}(Y)$ and transition matrix $B$ given by

$$B(b,b') = \begin{cases} 1 & \text{if } b_{[2,N]} = b'_{[1,N-1]},\ Z(b'_N) \cap \omega_+(Y,S_\Sigma)(b) \neq \emptyset, \\ 0 & \text{elsewhere}, \quad b,b' \in P_{[1,N]}(Y). \end{cases}$$

The mapping

$$x \to (x_{[i,i+N)})_{i \in \mathbf{Z}} \qquad (x \in Y)$$

is then a topological conjugacy of $(Y, S_\Sigma)$ onto $(X_B, S_{P_{[1,N]}(Y)})$. Q.e.d.

(4.2) LEMMA. *If $(Y, S_\Sigma)$ is properly sofic then there exists a periodic point $y \in Y$ such that*

$$|\rho_+^{-1}\{y\}| > 1.$$

PROOF. If $(Y, S_\Sigma)$ is properly sofic, then we have by Proposition (4.1) a point

$$x \in Y - C_+(Y, S_\Sigma). \tag{18}$$

Set

$$\Gamma_i = \bigcap_{j<i} \omega_+(Y,S_\Sigma)(Z(x_{[j,i)})), \qquad i \in \mathbf{Z}.$$

We claim that for all $i \in \mathbf{Z}$ the set $\Gamma_i$ contains more than one element. For a proof, assume that for some $i_0$, $\Gamma_{i_0}$ contains only one element. Then there exists a $j < i_0$ such that

$$|\{\omega_+(Y,S_\Sigma)(Z(x_{[j,i_0)}))| = 1.$$

This would mean that $x_{[j,i_0)}$ is a finitary block, a contradiction to (18).

We observe further that

$$|\Gamma_i| \leqq |\Gamma_{i+n}|, \qquad i \in \mathbf{Z}, \quad n \in \mathbf{N}.$$

In fact, it is

$$\Gamma_{i+n} = \{P_{[i+n,\infty)}(Z(x_{[i,i+n)}) \cap E) : E \in \Gamma_i\}.$$

Let now $i_1 \in \mathbf{Z}$ be such that

$$|\Gamma_{i_1+n}| = |\Gamma_{i_1}|, \qquad n \in \mathbf{N}.$$

There are $i_2, i_3 > i_1, i_2 < i_3$, such that

$$x_{i_2} = x_{i_3}, \qquad \Gamma_{i_2} = S_\Sigma^{i_2-i_3}\Gamma_{i_3}. \tag{19}$$

Let $y \in \Sigma^{\mathbf{Z}}$ be given by

$$y_{i_2+k(i_3-i_2)+l} = x_{i_2+l}, \qquad k \in \mathbf{Z}, \quad 0 \leqq l < i_3 - i_2.$$

It follows from (19) that $y \in Y$. Also by (19) we have for every $E \subset \Gamma_{i_2}$ an $M_E \in \mathbf{N}$ such that

$$E = S_\Sigma^{M_E} P_{[i_2+M_E,\infty)}(Z(x_{[i_2,i_2+M_E)}) \cap E),$$

and we produce for every $E \in \Gamma_{i_2}$ a point $(y_i, D_i)_{i \in \mathbf{Z}} \in X_+(Y, S_\Sigma)$ by setting

$$D_{i_2+kM_E+m} = S_\Sigma^m P_{[i_2+m,\infty)}(Z(x_{[i_2,i_2+m)}) \cap E), \quad k \in \mathbf{Z}, \quad 0 \leqq m < M_E.$$ Q.e.d.

(4.3) PROPOSITION. *The following are equivalent*:

(a) $(Y, S_\Sigma)$ *is properly sofic.*

(b) *There exists a non-F-finitary point.*

(c) *There exists a non-F-finitary periodic point.*

PROOF. Use Proposition (4.1) and Lemma (4.2) in conjunction with Lemma (2.6). Q.e.d.

(4.4) THEOREM. $(Y, S_\Sigma)$ *is Markov if and only if*

$$\zeta_{(Y,S_\Sigma)}(z) = \operatorname{Det}(1 - z\varphi_{(Y,S_\Sigma)})^{-1}. \tag{20}$$

PROOF. $\tau_+(Y, S_\Sigma)$ assigns to every periodic point of $(Y, S_\Sigma)$ a periodic point of $(X_+(Y, S_\Sigma), S_+(Y, S_\Sigma))$ with the same period. By Theorem (3.1) we have that (20) implies that all periodic points of $(X_+(Y, S_\Sigma), S_+(Y, S_\Sigma))$ are in the range of $\tau_+(Y, S_\Sigma)$.

Assume that $(Y, S_\Sigma)$ is properly sofic. Then we have by Lemmas (4.1) and (4.2) a periodic point $y$ of $(Y, S_\Sigma)$ that has under $\rho_+(Y, S_\Sigma)$ more than one inverse image. One of these inverse images is equal to $\tau_+(Y, S_\Sigma)(y)$, and the others cannot be in the range of $\tau_+(Y, S_\Sigma)$. Thus (20) cannot hold.

If $(Y, S_\Sigma)$ is Markov then (20) is the formula of Bowen–Lanford [2]. Q.e.d.

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